Chromatic Analysis of Numerical Program

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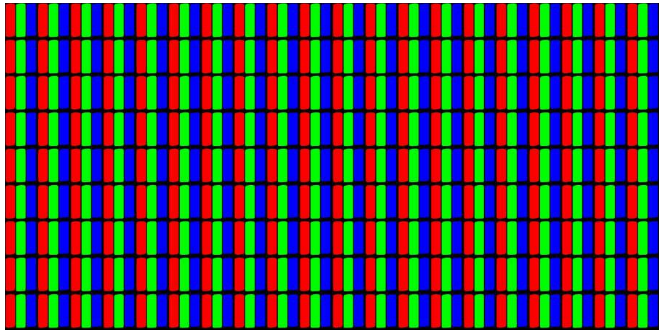


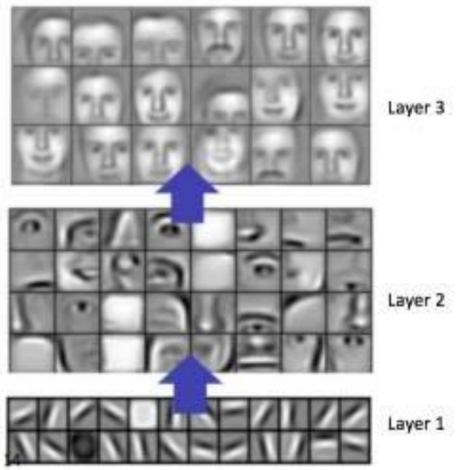
A few word about colors…. In RGB

RED (122,0,0)	+	BLUE (0,0,122)	=	PURPLE (122,0,122)	+	PURPLE (122,0,122)	=	PURPLE (254,0,254)
GREEN (0,122,0)	+	BLUE (0,0,122)	=	CIAN (0,122,122)	+	CIAN (0,122,122)	=	CIAN (0,254,254)
RED (122,0,0)	+	GREEN (0,122,0)	=	YELLOW (122,122,0)	+	YELLOW (122,122,0)	=	YELLOW (254,254,0)

Colors naturally provides visual information under **additive property**

Introduction

- Assessment
  - For some applications (DNN), we are more concerned by understanding the resulting value than by the propagation of errors
- Objective
  - Estimate the relations between input and output variables under additive property
- Solution
  - Use the concept of chromatic number to tint scalar or set of scalars
  - Each scalar is decomposed as the sum of tinted values



Chromatic number: Definition

- A **Chromatic Number** consists in associating a color to scalar or set of scalar in order to track them during computation
  - Corresponds to a triplet $\langle x, k_x, V_x \rangle$
    - x is the floating-point number
    - V_x is a vector of n floating-point numbers representing the weight of the n tints within x such that we have *Additive property*
    - $x = \frac{1}{k_x} \sum_{i=0}^{n} V_x[i]$
- Properties
  - V_x Corresponds to a component-wise decomposition of numerical values
  - Multiple scalars can be set with the same tint
    (track multiple values at the same time and helps to reduce the dimensionality of the problem)
  - Need to set a "Garbage element" in V_x to collect contributions of non-chromatic numbers to preserve additive property.
    (Optional element if every computation were done without rounding error ($x = \sum_{i=0}^{n} V_x[i]$))

Chromatic number: Operations

- Set a new arithmetic on chromatic numbers:
  - Addition: $< x, k_x, V_x > + < y, k_y, V_y > = < x + y , 1, \frac{V_x}{k_x} + \frac{V_y}{k_y} >$
  - Subtraction: $< x, k_x, V_x > - < y, k_y, V_y > = < x - y , 1, \frac{V_x}{k_x} - \frac{V_y}{k_y} >$
  - Multiplication: $< x, k_x, V_x > . < y, k_y, V_y > = < x.y , k_x + k_y, y.V_x + x.V_y >$
  - Division: $\frac{<x,k_x,V_x>}{<y,k_y,V_y>} = < \frac{x}{y}, \frac{\frac{x}{V_y}+\frac{V_x}{y}}{2} > = < \frac{x}{y} , k_x + k_y, \frac{x}{y^2}.V_y + \frac{V_x}{y} >$
  - Sqrt(x): $\sqrt{< x, k_x, V_x >} = < \sqrt{x}, 1, \frac{V_x}{k_x\sqrt{x}} >$
  - Any functions: $f(< x, k_x, V_x >, < y, k_y, V_y >) = < f(x + y), k_x + k_y, \frac{f(x,V_y)+f(V_x,y)}{2} >$

Example 1: Cancellation

- Let consider the sequence of operations
  - $a = 2$
  - $b = 3$
  - $r = (a.a).b - 12$

- In CA this corresponds to
  - $a =< 2, 1, [0,2,0] >$ b$=< 3, 1, [0,0,3] >$

    Garbage element

  - $(a.a) =< 4, 2, [0,8,0] >$
  - $(a.a).b =< 12, 3, [0,24,12] >$
  - $(a.a).b - 12 =< \mathbf{0}, 1, [-12, 8, 4] >$

Example 2: LP Digital Filter

- Goal:
  - Understand how output results are affected by input data, program parameters, etc over time
  - Relative weight in the **output value** of the **input values**

```python
import numpy as np
from scipy.signal import butter, lfilter, freqz
import matplotlib.pyplot as plt

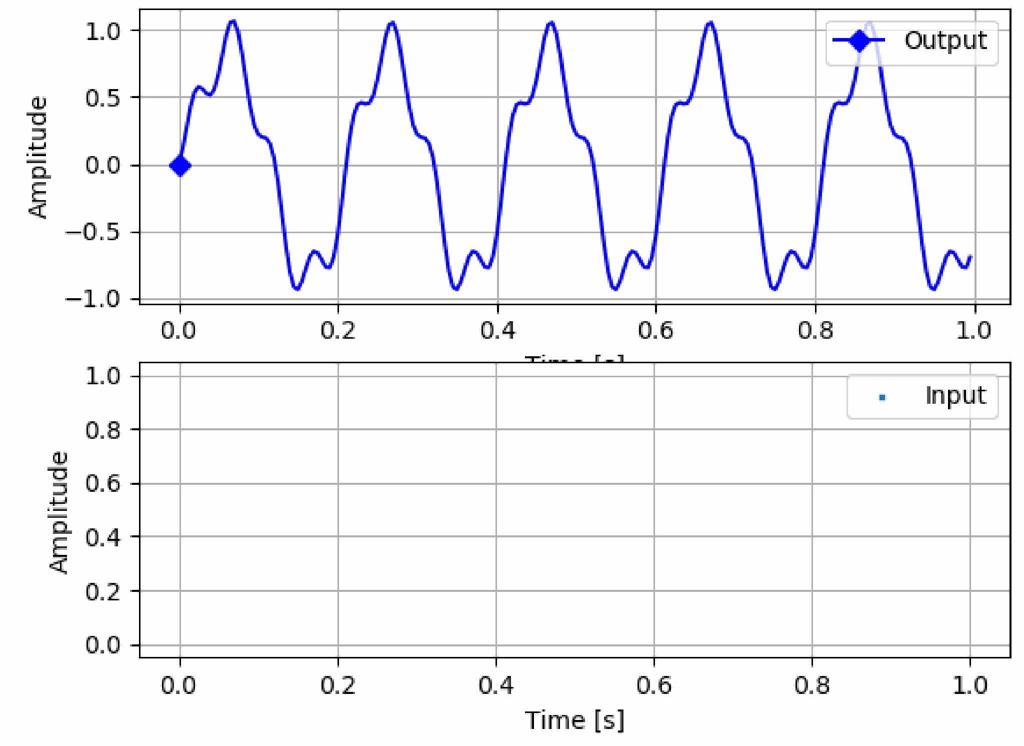
# Create a low-pass Butterworth filter
def butter_lowpass(cutoff, fs, order=5):
    nyquist = 0.5 * fs
    normal_cutoff = cutoff / nyquist
    b, a = butter(order, normal_cutoff, btype='low', analog=False)
    return b, a

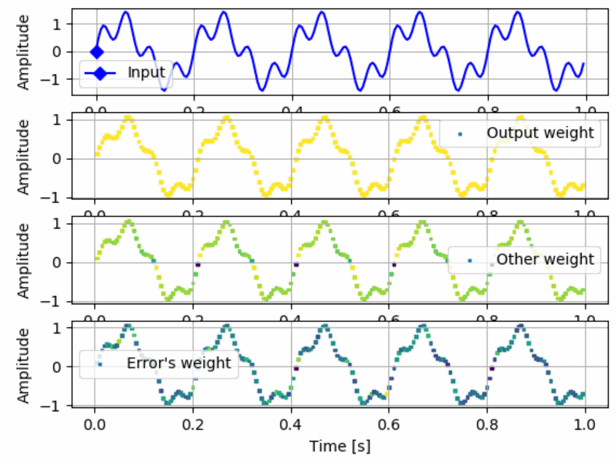
# Apply the filter to the input signal
def butter_lowpass_filter(data, cutoff, fs, order=5):
    b, a = butter_lowpass(cutoff, fs, order=order)
    y = lfilter(b, a, data)
    return y

# Example usage
# Generate some random input data
fs = 100.0       # Sample rate (Hz)
t = np.linspace(0, 1, int(fs), endpoint=False)
data = np.sin(2 * np.pi * 5 * t) + 0.5 * np.sin(2 * np.pi * 20 * t)

# Filter parameters
order = 6
cutoff_freq = 10.0   # Desired cutoff frequency (Hz)

# Apply the filter to the input data
filtered_data = butter_lowpass_filter(data, cutoff_freq, fs, order)
```

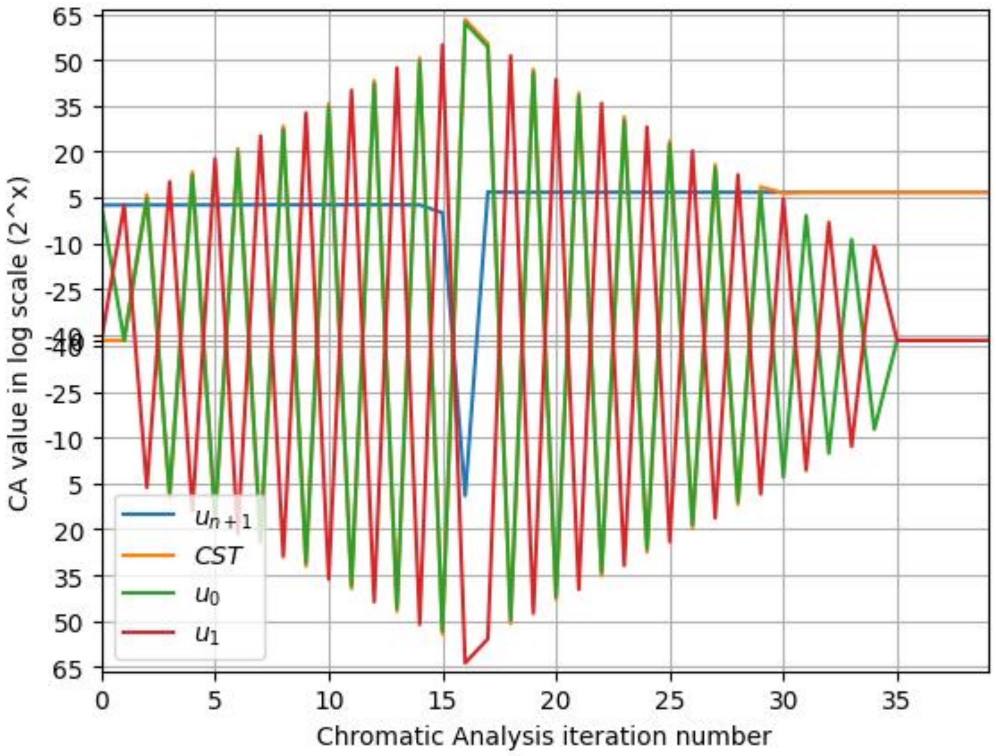


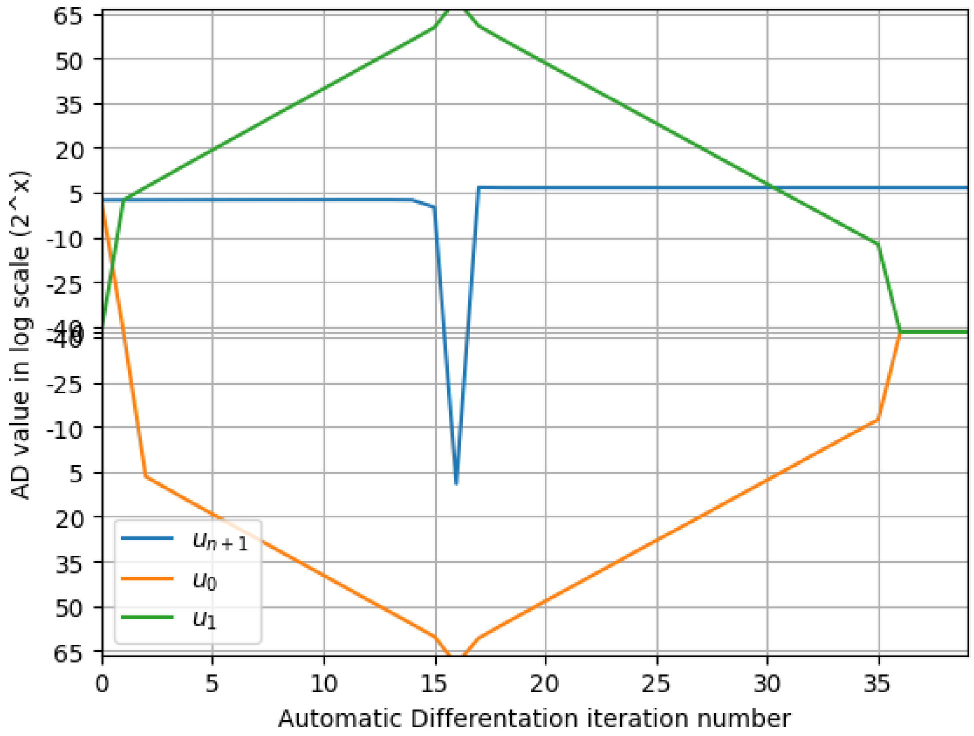


Example 3: Muller's series

$$\begin{cases} u_0 & = & 5.5 \\ u_1 & = & 61.0/11.0 \\ u_{n+1} & = & 111. - 1130./u_n + 3000./(u_n.u_{n-1}) \end{cases}$$

- Goal:
  - Give a fine interpretation of what is numerically happening in some pathologicals cases





Related works

1. Sensitivity analysis

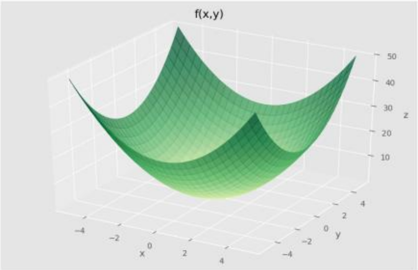
- Evaluate how variations in input parameters affect the output
- Identify which input parameters have the greatest effect on the output
- Issues:
  - Curse of dimensionality, inability to handle correlated input, difficult to interpret variation on multiple input

2. Automatic Differentiation

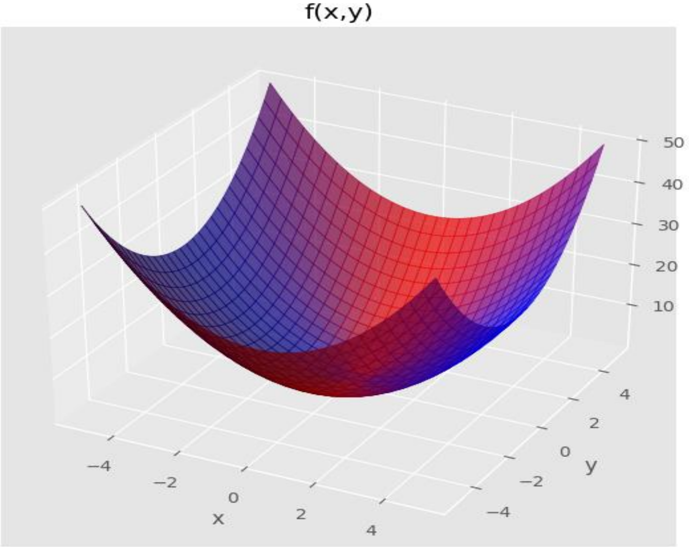
- Compute the gradient at each step
- Forward or backward according to the input/output dimensionality
- Implementation:
  - Each number X is replaced by a Dual Number $\langle x|x'\rangle$ where x' is the derivative such that $X = x + x'\varepsilon$ with ε an abstract number such that $\varepsilon^2 = 0$.
- Problematic with complex multivariate program as untracked variables are not taken into account
  (Solution: rely on automated sparsity detection of the Jacobian matrix)

Graphical illustration of CA vs AD

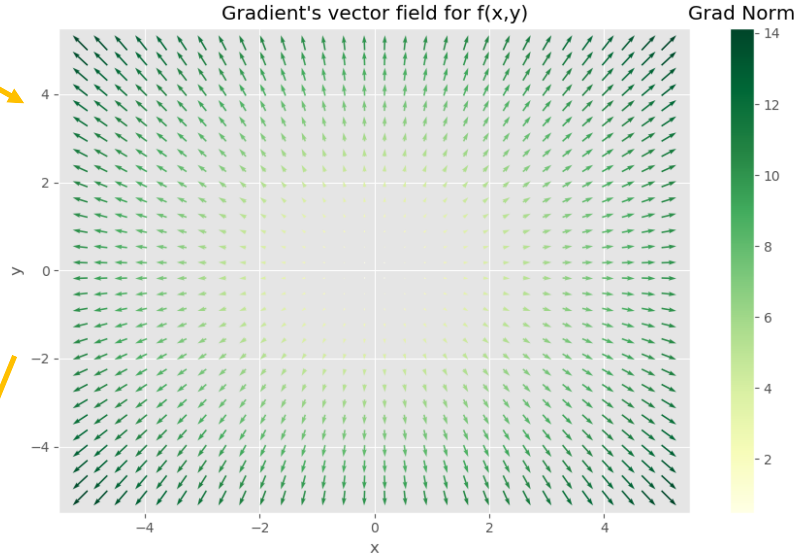
$$f(x, y) = x^2 + y^2$$

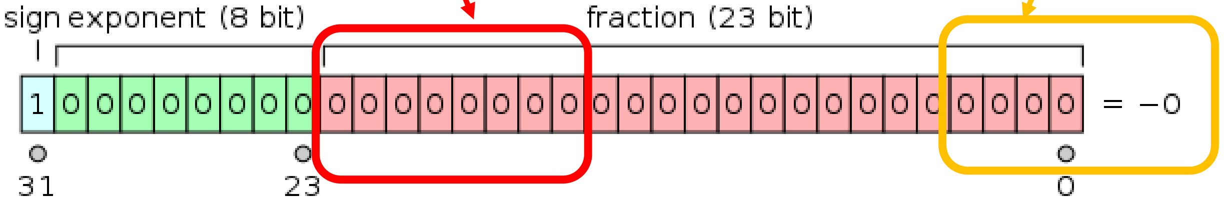


Relative weight of x (red) and y (blue) in f(x,y)



Local steepness measure

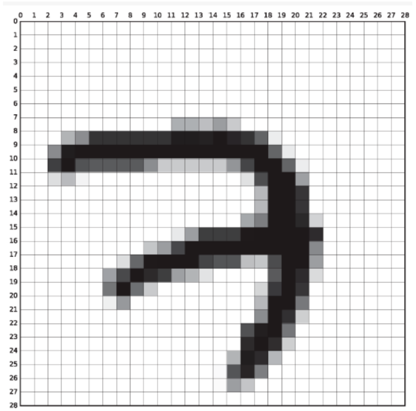




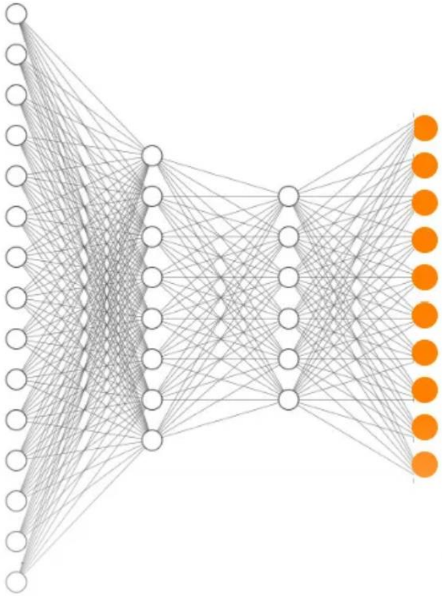
Chromatic analysis

AD analysis

Experiments: inference in DNN MNIST



1 pixel = 1 index

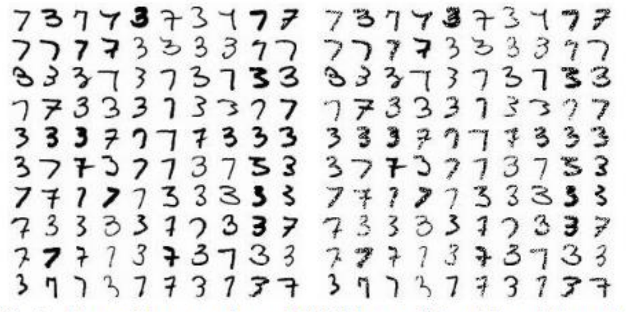


784,100,50,10 Network

P0=0.001	<0.0,-0.010,0.020, ….,0.0>
P1=0.001	<0.0,0.120,0.085, ….,0.0>
P2=0.003	<0.0,0.037,-0.008, ….,0.0>
P3=0.005	<0.0,-0.062,-0.011, ….,0.0>
P4=0.005	<0.0,0.074,0.001, ….,0.0>
P5=0.020	…
P6=0.010	
P7=0.950	
P8=0.002	<0.0,0.003,-0.007, ….,0.0>
P9=0.003	<0.0,-0.003,-0.008, ….,0.0>

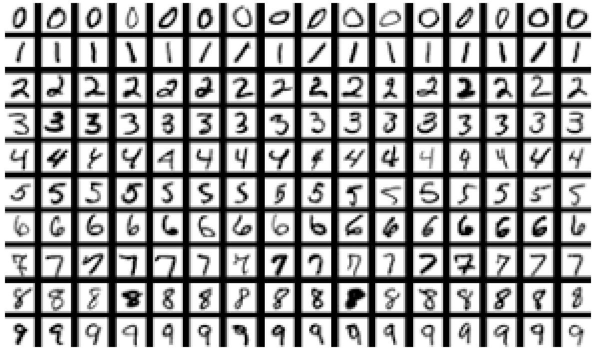
Output probability +
Pixel-wise decomposition
(784+1 elements)

Possible usage: adversarial attack to alter output probability
classification by minimizing the number of modified pixel
Execution overhead: x100 on time, x20 on memory

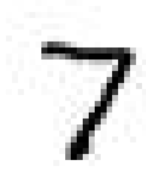


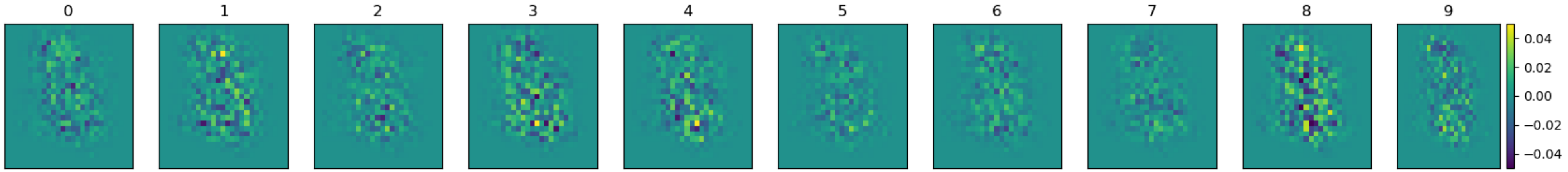
Experiments: Training DNN MNIST

- Goal
  - Track the weight of image class during learning phase
  - Understand the network's numerical behavior
- Methodology
  - Tint according to its image classification (0 to 9)
  - Each of the 84k coefficients are decomposed according to image classification
- Ouput
  - Resulting network made of coefficient decomposed as chromatic numbers tinted according to the input images class
- Overhead
  - x1800 time (not possible to use optimized BLASS)
  - x12 on memory



For an image:
each pixel = same index





Conclusion

- Chromatic analysis
  - Additive decomposition of results according to tinted values/set of values
  - Allows **fusion of data** to limit the dimensionality problem encountered with other analysis
  - Helps understand what is important among input values, constant, scalar and their numerical relation (ex: cancellations)
  - Preserve the input data structures

- Future works
  - Thanks to the additive property, it is possible to combine this analyse with an iterative refinement algorithm to reduce the memory overhead
    - Start with a few sets of tracked values (low memory / low computational overhead),
    - Restart the analyses by splitting only sets of variables which are having a large impact
  - Combine chromatic analysis with others to reduce the cost of global sensitivity analysis
    - Will help focusing on variables of interest.
  - Investigate various tinting mechanism
    - According to data type, time, location (functions, MPI Process…)
  - Test on real life program (numerically instruct abnormality)

Example: Error Free Transformation

(s , t) = Fast2Sum(a , b) // a>>b
s = a + b
r = s – a
t = b – r

a = <a , 1 , [0 , a , 0]>
b = <b , 1 , [0 , 0 , b]>

s = <a , 1 , [0 , a , b]>
r = <0 , 1 , [0 , 0 , b]>
t = <b , 1 , [0 , 0 , 0]>

$$0 \neq 0 + 0 + b$$

Does not take into account rounding error

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$$0 \neq 0 + 0 + b$$

Does not take into account rounding error
=> Include a rounding error term
=> Dedicated routines for EFT

a = <a , 1 , [0 , 0 , a , 0]>
b = <b , 1 , [0 , 0 , 0 , b]>

s = <a , 1 , [0 , -b , a , b]>
r = <0 , 1 , [0 , -b , 0 , b]>
t = <b , 1 , [0 , b , 0 , 0]>

Chromatic number: Implementation

- Space and time complexity grows linearly with the number of tinted values.
  - Example: A chromatic analysis on a 8 Mb dense matrix will lead to 8 Tb of intermediate representation.
  - C++/Python implementation with V_x stored either as a dense/sparse structure (vector/dictionary)
- Optimization: Fusion of small contributions
  - Discard tinted element which are becoming too small compared to others and accumulate them in the garbage element ($\left|\frac{V_{x[i]}}{V_{x[j]}}\right| \geq C$ with C a tunable parameter typically set to 2^{53} for double precision). Particularly useful when used when V_x is a dictionary structure.
- Optimization: Error element
  - Set an element to track rounding errors performed on x in $\langle x|V_x\rangle$
  - Accumulate rounding error similarly to compensated algorithm (use of EFT & extended precision)

Chromatic number: Implementation

- Optimization: Refinement algorithm
  - Start the chromatic analysis by aggregating the maximum number of value under the same tint in order to minimize the size of V_x .
  - Detect which tint account for the most and restart the computation by subdividing the selected tint, while detecting under-approximation (cancellation within a tint)

Algorithm 1 Contribution refinement subdivision algorithm

Require: *O=func(I)* the function to analyse
Require: I the set of scalar to track
Require: $card(I) = N$, and $O = \langle o, V_o \rangle$

  I'=split(I) ▷ Initial Spliting
  do
    O'=func(I')
    S =False
    for i in I' **do**
      if $|o'| > k_0|V_{o'}[1]|$ and $|V_{o'}[i+2]| > k_1|V_{o'}[1]|$ and card(I'[i])>1 **then**
        I'=split(I'[i])
        S =True
      end if
    end for
  while S

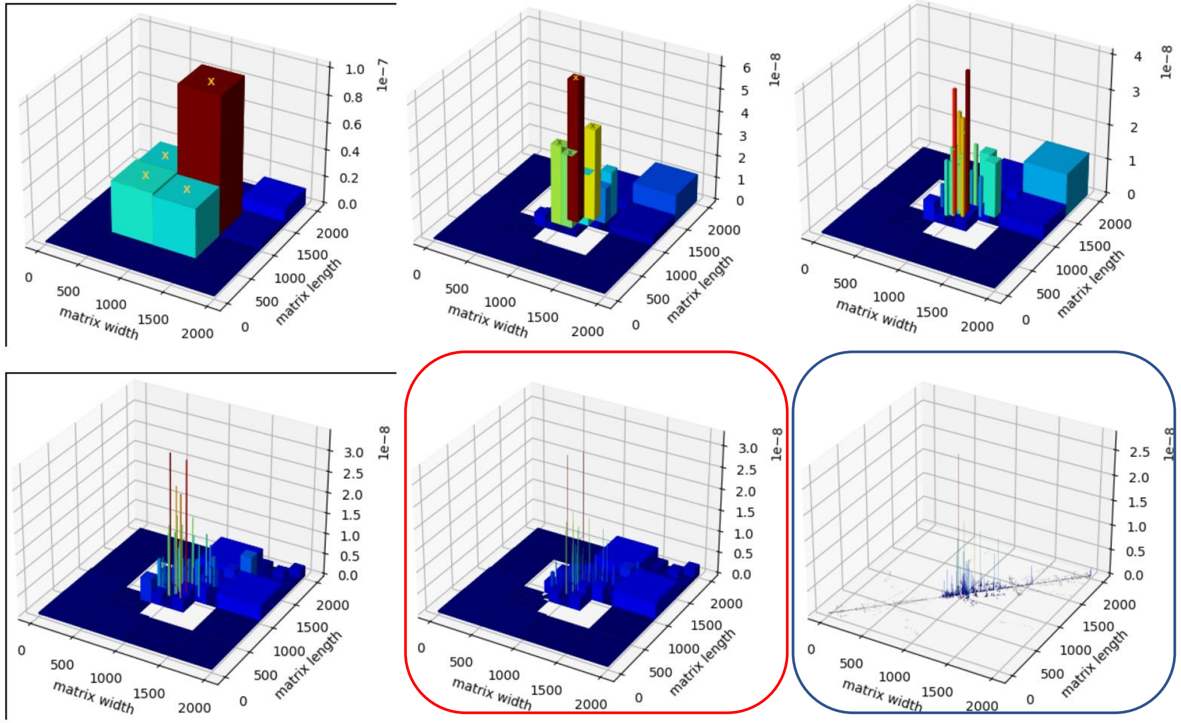
Experiment : Sparse solver

- Matrix from MatrixMarket:
  - BCSSTK13: size 2003 x 2003; 42943 entries; estimated conditioned number 4.6 1010
  - BCSSTK14: size 1806 x 1806; 32630 entries; estimated conditioned number 1.3 1010
- Execution time in sec. and memory to solve BCSSTK14 between Python and C++ version.
  - 6-10x overhead in Python, 10-700x overhead in C++ (due to the sparsity of the system)
  - Memory usage grows linearly => x500-1000 on memory for 1000 tinted values

Number of tinted value followed	*no-instr*	1	16	32
Python	250s /70Mo	1634s /108Mo	2022s /125Mo	2500s /156Mo
C++	0.13s /21Mo	1.3s /26Mo	40s /135Mo	92s /253Mo

Experiment N°3: Sparse solver

- Iterative refinement algorithm , starting with a 4x4 subdivision according to the index in each direction of the matrix BCSSTK13.
- Stops after 5 iterations in 836 sec.



Reference
Analysis conduced while keeping the 128 most contributing tint in each cell. (2205 sec)
=> More time consuming and less precise than the iterative algorithm

“A picture is worth a thousand words”

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import matplotlib.pyplot as plt

# Create a low-pass Butterworth filter
def butter_lowpass(cutoff, fs, order=5):
    nyquist = 0.5 * fs
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# Apply the filter to the input signal
def butter_lowpass_filter(data, cutoff, fs, order=5):
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# Generate some random input data
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t = np.linspace(0, 1, int(fs), endpoint=False)
data = np.sin(2 * np.pi * 5 * t) + 0.5 * np.sin(2 * np.pi * 20 * t)

# Filter parameters
order = 6
cutoff_freq = 10.0   # Desired cutoff frequency (Hz)

# Apply the filter to the input data
filtered_data = butter_lowpass_filter(data, cutoff_freq, fs, order)
```