Using Hierarchical Approach to Speed-up RNS Base Extensions in Homomorphic Encryption Context

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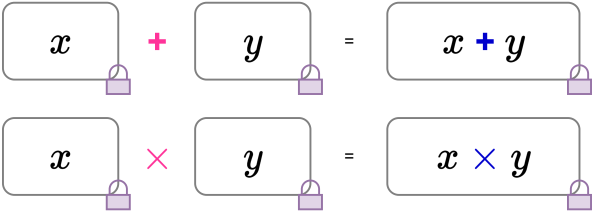
Homomorphic Encryption [Gen09]

Homomorphic encryption (HE) allows to perform computations over encrypted data without the knowledge of the secret key

Solution for privacy-concerned applications: medical/machine learning



Huge cost: requires numerous arithmetic operations on very large intermediate data

$\Rightarrow$ Arithmetic is key for HE since it requires operations on very large data

Chosen HE solution: FV scheme [FV12]

The FV scheme handles huge polynomials:

- degree $n \in [2^{11}, 2^{17}]$
- with large coefficients in $\mathbb{Z}/q\mathbb{Z}$ with $\log_2(q) \in [55, 2090]$ bits

$$c = \boxed{c_0} + \boxed{c_1}\, x + \boxed{c_2}\, x^2 + \ldots + \boxed{c_n}\, x^n$$

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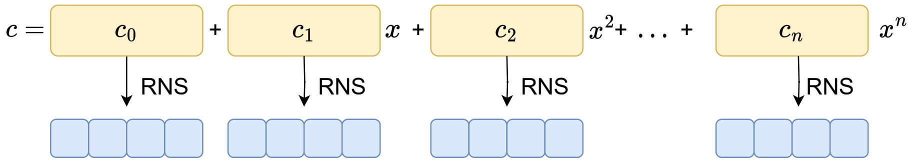


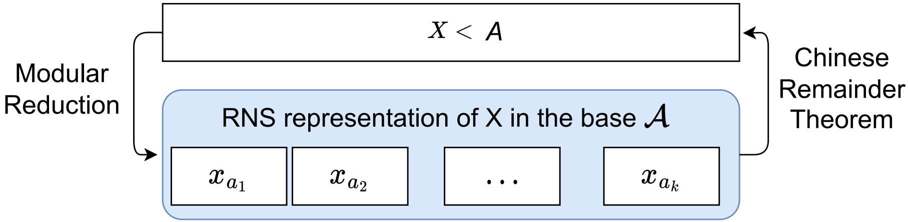
RNS is used for the arithmetic on coefficients

Residue Number System (RNS) [Gar59] [SV55]

RNS is a non-positional representation system based on the Chinese Remainder Theorem (CRT)

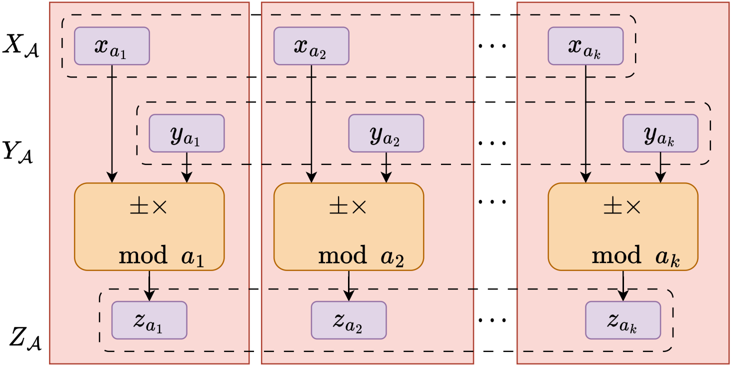
RNS Base: $\mathcal{A} = (a_1, \ldots, a_k)$ tuple of coprime integers of size w bits called **moduli**, $\prod_{i=1}^{k} a_i = A$

Representation of an integer X in the base $\mathcal{A}$:
$X_{\mathcal{A}} = (|X|_{a_1}, \ldots, |X|_{a_k}) = (x_{a_1}, \ldots, x_{a_k})$ where $|X|_{a_i} = X \bmod a_i$



Pros and cons

Pros: Operations $+, -, \times$ computed independently over the residues
$\sim$ **only k small operations over the residues**
$\Rightarrow$ fast and parallel



Cons: Division, rounding and modular reduction are difficult to compute in RNS $\Rightarrow$ intermediate operation: Base Extension

Base Extension

A base extension (BE) converts X in base $\mathcal{A}$ into X in base $\mathcal{B}$
BE costs $\boldsymbol{\sim k^2}$ operations

Most popular technique: computing the CRT in each moduli of the second base

CRT formula:

$$|X|_{b_j} = \left| \left| \sum_{i=1}^{k} \left| x_{a_i} \cdot \frac{a_i}{A} \right|_{a_i} \cdot \frac{A}{a_i} \right|_A \right|_{b_j} = \left| \left(\sum_{i=1}^{k} \left| x_{a_i} \cdot \frac{a_i}{A} \right|_{a_i} \cdot \frac{A}{a_i} \right) - hA \right|_{b_j}$$

Popular BE algorithms:

- Kawamura and al. [KKSS00] (approximate computation of h)
- Shenoy and Kumaresan [SK89] (requires specific conditions)

FV in RNS

Adaptations of FV in RNS:

- first adaptation of FV fully in RNS was proposed in [BEHZ16]
  - speed-up from 5 to 20 for the decryption
  - speed-up from 2 to 4 for the homomorphic multiplication
- variants in [HPS19] [ABVMA18] [ABPA$^+$21] [KPZ21]

Difficult operations:

- roundings
- modular reduction

In the state-of-the-art, half of the homomorphic multiplication is spent doing base extension

Fast Base Extension (FBE)

[BEHZ16] proposed to compute these BE with a Fast Base Extension algorithm (FBE)

$$\mathrm{FBE}(X, \mathcal{A}, \mathcal{B}) = \left| \left(\sum_{i=1}^{k} \left| x_{a_i} \cdot \frac{a_i}{A} \right|_{a_i} \cdot \frac{A}{a_i} \right) \right|_{b_j}$$

- Efficient because it removes the costly modular reduction by A in the CRT formula but gives an approximate result
- [BEHZ16] manages the approximation with some correction steps outside the BE

Hierarchical Base Extension (HBE)

[DBT19] proposed the hierarchical base extension: a hierarchical version of Kawamura Base Extension to speed-up ECC on FPGA

New notation: $\mathcal{A} = (a_1, \ldots, a_k) \longrightarrow \mathcal{A} = \begin{pmatrix} a_{1,1} & \ldots & a_{1,c} \\ \vdots & \ddots & \vdots \\ a_{r,1} & \ldots & a_{r,c} \end{pmatrix}$

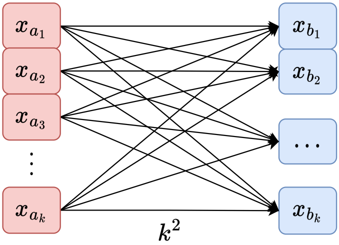
with $k = r \times c$

Idea:

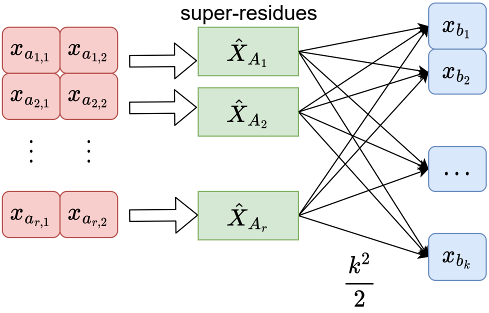
- split the base of k moduli in r sub-bases of c moduli
- compute the CRT in each sub-base to create super-residues
- compute the CRT of the super-residues of base $\mathcal{A}$ in base $\mathcal{B}$

In [DBT19], $c = 2$

Kawamura Base Extension:



HBE: $k = r \times c$ with $c = 2$



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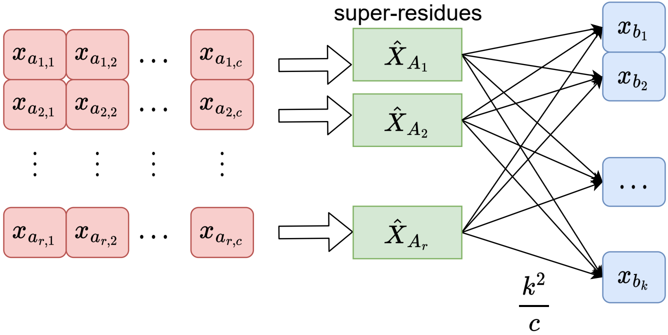
Proposed Fast Hierarchical Base Extension (FHBE)

We adapt the hierarchical approach to FBE for software implementation of FV

We study the impact of the value of c on the computation cost using prime moduli



$r = \frac{k}{c}$

Algorithm 1: FBE [BEHZ16]

Input: $X_{\mathcal{A}}$

Precomp.: $T_{a_{i,j}}$, $\forall\ i \in [1, r]$ and $\forall\ j \in [1, c]$, $\left|\frac{A}{a_i}\right|_{b_{l,j}}$ $\forall\ i \in [1, k],\ l \in [1, r],\ j \in [1, c]$

Output: $X + \alpha A$ in base $\mathcal{B}$

for i **from** 1 **to** r **parallel do**
    for j **from** 1 **to** c **parallel do**
        $\widehat{x}_{a_{i,j}} \leftarrow \left|x_{a_{i,j}} \times T_{a_{i,j}}\right|_{a_{i,j}}$
for i **from** 1 **to** k **do**
    for l **from** 1 **to** r **parallel do**
        for j **from** 1 **to** c **parallel do**
            $x_{b_{l,j}} \leftarrow \left|x_{b_{l,j}} + \widehat{x}_{a_i} \times \left|\frac{A}{a_i}\right|_{b_{l,j}}\right|_{b_{l,j}}$

Algorithm 2: Proposed Fast Hierarchical BE (FHBE)

Input: $X_{\mathcal{A}}$

Precomp.: $T_{a_{i,j}}$, $\overline{a_{i,j}}$ $\forall\ i \in [1, r]$ and $\forall\ j \in [1, c]$, $\left|\overline{A_i}\right|_{b_{l,j}}$ $\forall\ i \in [1, r]$, $\forall\ l \in [1, r]$ and $\forall\ j \in [1, c]$

Output: $X + \alpha A$ in base $\mathcal{B}$

for i **from** 1 **to** r **parallel do**
    for j **from** 1 **to** c **parallel do**
        $\widehat{x}_{a_{i,j}} \leftarrow \left|x_{a_{i,j}} \times T_{a_{i,j}}\right|_{a_{i,j}}$
for i **from** 1 **to** r **parallel do**
    $\widehat{X}_{A_i} \leftarrow 0$
    for j **from** *1* **to** c **do**
        $\widehat{X}_{A_i} \leftarrow \widehat{X}_{A_i} + \widehat{x}_{a_{i,j}} \times \overline{a_{i,j}}$ *(no reduction)*
for i **from** 1 **to** r **do**
    for l **from** 1 **to** r **parallel do**
        for j **from** 1 **to** c **parallel do**
            $\widehat{x}_{b_{l,j,i}} \leftarrow \left|\widehat{X}_{A_i}\right|_{b_{l,j}}$
            $x_{b_{l,j}} \leftarrow \left|x_{b_{l,j}} + \widehat{x}_{b_{l,j,i}} \times \left|\overline{A_i}\right|_{b_{l,j}}\right|_{b_{l,j}}$

Cost Comparisons

FBE:

Number of operations:

- $k^2 + k$ modular multiplications on w bits

Stored precomputations:

- $k^2 + k$ stored precomputations of size w bits

FHBE:

Number of operations:

- $\frac{k^2}{c} + k$ modular multiplications on w bits
- $w \times (c-1)w$ bits multiplications without reduction
- modular reductions from $cw + \lceil \log_2(c) \rceil$ to w bits

Stored precomputations:

- $\frac{k^2}{c} + k$ stored precomputations of size w bits

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Implementation

Implementation of our FHBE and the FBE algorithms:

- in C language using the GMP multiple-precision arithmetic library version 6.2.0
- GCC compiler version 9.4.0
- Linux Kernel 5.15 from Ubuntu distribution

Performance and memory cost evaluations performed on a Intel Core i7-9850H processor at 2.60GHz

We analyzed:

- the computation cost using the execution time on a single thread
- the pre-computation storage requirements

Evaluation Cases

To use FHBE, k must be divisible ($k = r \times c$)

This leads to two cases:

- k from [BEHZ16] is **divisible**: we use k for both FBE and FHBE
- k from [BEHZ16] is **prime**
  - we use k for FBE
  - we choose a close k with many divisors for FHBE (with the same security in number of bits)

Results when k is divisible

k: number of moduli
w: size of the moduli in bits
$\log_2(q) = k \times w$

The parameters (k, w) come from [BEHZ16]

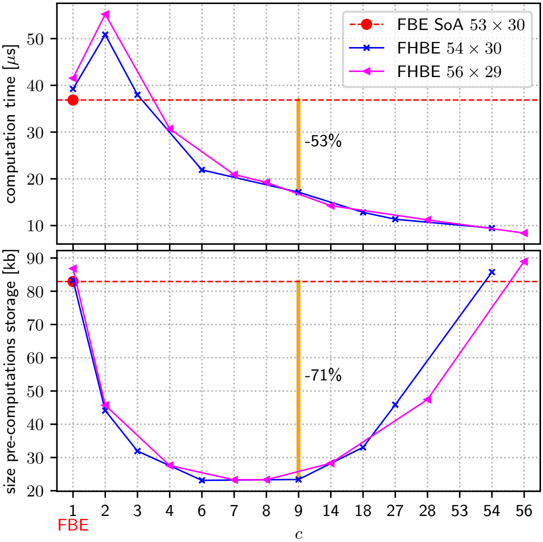
	$k \times w$	FBE	FHBE with c on line below						best gain
			2	3	4	5	6	13	
time [μs]	12×62	2.74	3.47	2.72	2.35	-	**1.96**	-	28%
	26×30	10.21	12.67	-	-	-	-	**4.26**	58%
	25×62	9.53	-	-	-	**6.61**	-	-	30%
size [kb]	12×62	9.53	5.86	**5.14**	5.15	-	5.92		46%
	26×30	20.32	**11.38**	-	-	-	-	11.6	43%
	25×62	39.65	-	-	-	**15.33**	-	-	61%

Results when k is prime

In [BEHZ16], $(k, w) = (53, 30)$ is proposed

53 is prime, then we use:

- $(54, 30)$ with $54 = 2 \times 3^3$
- $(56, 29)$ with $56 = 2^3 \times 7$



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Conclusion

FHBE, a hierarchical variant of the FBE algorithm has been proposed and implemented for software RNS implementations of the FV scheme

It reduces:

- up to 58% the computation cost
- up to 71% the storage requirement

Future prospect:

- optimized multi-core implementation
- complete homomorphic library

Thank you for your attention

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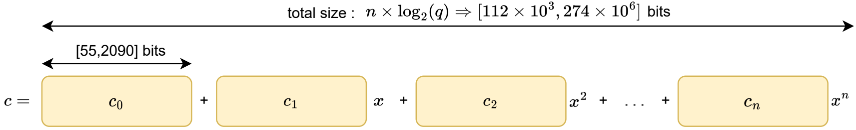
One popular HE solution: FV scheme [FV12]

FV: HE scheme based on the ring learning with error problem [LPR10]

Plain data: 1 polynomial of degree $n \in \left[2^{11}, 2^{17}\right]$ with coefficients in $\mathbb{Z}/t\mathbb{Z}$ with $t \geq 2$
Cipher data: 2 polynomials of degree $n \in \left[2^{11}, 2^{17}\right]$ with coefficients in $\mathbb{Z}/q\mathbb{Z}$ with $\log_2(q) \in [55, 2090]$ and $t < q$



For: 256 B < message < 16 kB ⇒ 28 kB < cipher < 68 MB

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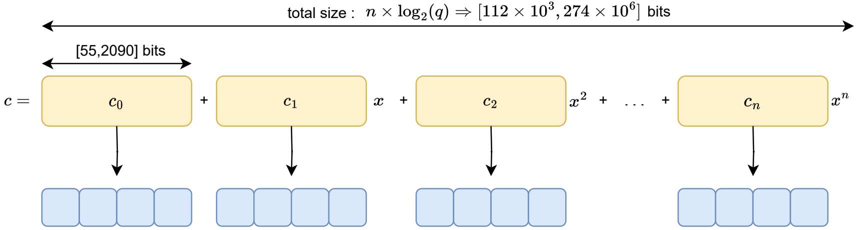

For: 256 B $<$ message $<$ 16 kB $\Rightarrow$ 28 kB $<$ cipher $<$ 68 MB

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- Efficient because it removes the costly modular reduction by A in the CRT formula but gives an approximate result
- [BEHZ16] manages the approximation with some correction steps outside the BE

Cost of the BE for an homomorphic multiplication in FV RNS (implementation results from [HPS19])